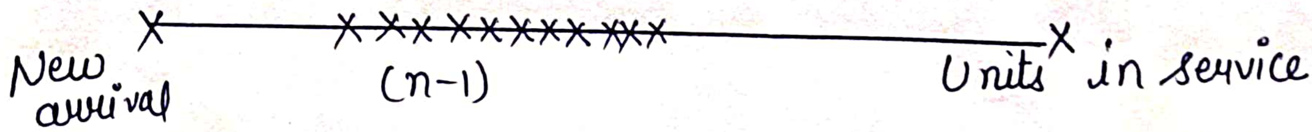
Class - Msc. (Mathematics)

Subject - operation Research-II

Topic :- The Distribution of Waiting time.

Let w denote the waiting time of an arrival in the system before it is taken into service. Evidently, it will be a random variable as its value depends upon the no. of units already waiting in the system and the time taken for service by each unit.

Let $P_n(w)$ denote the prob. that the waiting time of any particular unit lies between w and $w+dw$. when there are already n units in the queue and it joined the system out of these n units, $(n-1)$ Units are waiting and 1 in service as shown below:



The distribution of w can be divided into two parts.

1. Discrete distribution when $n=0$ i.e when the system is empty and the new arrival is immediately taken into service, then its waiting time is 0

$$\therefore \; P(w=0) = P_0 = 1-\rho$$

2. Continuous distribution when queue length is positive

If there are already n units in the system then a new arrival has to wait by time betweem w and $w+dw$. when $(n-1)$ units are serviced in time w and 1 unit in service is completely served in time dw.

Since the service distribution is Poisson

∴ Prob. of any unit in service at the time of new arrival is serviced in the time dw is μdw

The prob. of remaining $(n-1)$ units is given by

$$\frac{(\mu w)^{n-1} e^{-\mu w}}{\lfloor n-1}$$

∴ $P_n(w)\,dw$ = Prob. that new arrival is taken into service after a time between w and $w+dw$ when there are already n units in the system.

$P_n(w)\,dw$ = (Prob. that 1 unit is being serviced in dw) × (Prob. that $(n-1)$ units are served in time w)

$$= \frac{\mu\, dw \times (\mu w)^{n-1} e^{-\mu w}}{\lfloor n-1}$$

Now the queue length can vary from 1 to ∞, this is prob dist. of the waiting time is given by

$$P(w)dw = \sum_{n=1}^{\infty} P_n(w)dw \times (\text{Prob. that there are } n \text{ units in the system})$$

$$P(w)\,dw = \sum_{n=1}^{\infty} P_n(w)dw \times P_n$$

$$P(w)dw = \sum_{n=1}^{\infty} \mu e^{-\mu w} \frac{(\mu w)^{n-1}}{\lfloor n-1} (1-\rho)\rho^n dw$$

$$P(w)dw = \mu\rho(1-\rho)\, e^{-\mu w} \sum_{n=1}^{\infty} \frac{(\rho\mu w)^{n-1}}{\lfloor n-1} dw$$

$$P(w)dw = \mu\rho(1-\rho)\, e^{-\mu w}\, e^{\rho\mu w} dw$$

$$P(w)\,dw = \mu\rho(1-\rho)\, e^{-\mu w(1-\rho)} dw$$

Thus the Complete prob. distribution of the waiting time is given by

$$P(w) = \begin{cases} 1-\rho, & w = 0 \\ \mu\rho(1-\rho)\, e^{-\mu w(1-\rho)}, & w > 0 \end{cases}$$